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# Convex hulls of random walks and their scaling limits

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## Abstract

For the perimeter length and the area of the convex hull of the first  $n$  steps of a planar random walk, we study  $n \rightarrow \infty$  mean and variance asymptotics and establish non-Gaussian distributional limits. Our results apply to random walks with drift (for the area) and walks with no drift (for both area and perimeter length) under mild moments assumptions on the increments. These results complement and contrast with previous work which showed that the perimeter length in the case with drift satisfies a central limit theorem. We deduce these results from weak convergence statements for the convex hulls of random walks to scaling limits defined in terms of convex hulls of certain Brownian motions. We give bounds that confirm that the limiting variances in our results are non-zero.

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## 1. Introduction

Random walks are classical objects in probability theory. Recent attention has focused on various geometrical aspects of random walk trajectories. Many of the questions of stochastic geometry, traditionally concerned with functionals of independent random points, are also of interest for point sets generated by random walks. Here we examine the asymptotic behaviour of

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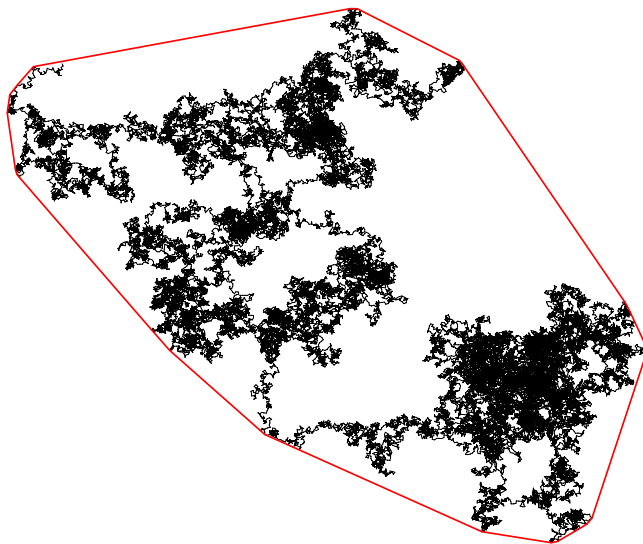


Fig. 1. Simulated path of a zero-drift random walk and its convex hull.

the *convex hull* of the first  $n$  steps of a random walk in  $\mathbb{R}^2$ , a natural geometrical characteristic of the process. Study of the convex hull of planar random walk goes back to Spitzer and Widom [21] and the continuum analogue, convex hull of planar Brownian motion, to Lévy [15, §52.6, pp. 254–256]; both have received renewed interest recently, in part motivated by applications arising for example in modelling the ‘home range’ of animals. See [16] for a recent survey of motivation and previous work. The method of the present paper in part relies on an analysis of *scaling limits*, and thus links the discrete and continuum settings.

Let  $Z$  be a random vector in  $\mathbb{R}^2$ , and let  $Z_1, Z_2, \dots$  be independent copies of  $Z$ . Set  $S_0 := 0$  and  $S_n := \sum_{k=1}^n Z_k$ ;  $S_n$  is the planar random walk, started at the origin, with increments distributed as  $Z$ . We will impose a moments condition of the following form:

( $M_p$ ) Suppose that  $\mathbb{E}[\|Z\|^p] < \infty$ .

Throughout the paper we assume (usually tacitly) that the  $p = 2$  case of ( $M_p$ ) holds. For several of our results we impose a stronger condition and assume that ( $M_p$ ) holds for some  $p > 2$ , in which case we say so explicitly.

Given ( $M_p$ ) holds for some  $p \geq 2$ , both  $\mu := \mathbb{E}Z \in \mathbb{R}^2$ , the mean drift vector of the walk, and  $\Sigma := \mathbb{E}[(Z - \mu)(Z - \mu)^\top]$ , the covariance matrix associated with  $Z$ , are well defined;  $\Sigma$  is positive semidefinite and symmetric. We also write  $\sigma^2 := \text{tr } \Sigma = \mathbb{E}[\|Z - \mu\|^2]$ . Here and elsewhere  $Z$  and  $\mu$  are viewed as column vectors, and  $\|\cdot\|$  is the Euclidean norm.

For a subset  $\mathcal{S}$  of  $\mathbb{R}^d$ , its convex hull, which we denote  $\text{hull } \mathcal{S}$ , is the smallest convex set that contains  $\mathcal{S}$ . We are interested in  $\text{hull } \{S_0, S_1, \dots, S_n\}$ , which is a (random) convex polygon, and in particular in its perimeter length  $L_n$  and area  $A_n$ . (See Fig. 1.)

The perimeter length  $L_n$  has received some attention in the literature, initiated by the remarkable formula of Spitzer and Widom [21], which states that

$$\mathbb{E}L_n = 2 \sum_{k=1}^n k^{-1} \mathbb{E}\|S_k\|, \quad \text{for all } n \in \mathbb{N} := \{1, 2, \dots\}. \quad (1)$$

Table 1

Each of the simulation estimates is based on  $10^5$  instances of a walk of length  $n = 10^5$ . The final decimal digit in each of the numerical upper (lower) bounds has been rounded up (down).

	Lower bound	Simulation estimate	Upper bound
$u_0(I)$	$2.65 \times 10^{-3}$	1.08	9.87
$v_0$	$8.15 \times 10^{-7}$	0.30	5.22
$v_+$	$1.44 \times 10^{-6}$	0.019	2.08

Much later, Snyder and Steele [20] obtained the law of large numbers  $\lim_{n \rightarrow \infty} n^{-1} L_n = 2\|\mu\|$ , a.s.; this is stated for the case  $\mu \neq 0$  in [20] but the proof works equally well in the case  $\mu = 0$ . To prove their law of large numbers, Snyder and Steele used the Spitzer–Widom formula (1) and the variance bound [20, Theorem 2.3]

$$n^{-1} \mathbb{V} \text{ar } L_n \leq \frac{\pi^2 \sigma^2}{2}, \quad \text{for all } n \in \mathbb{N}. \quad (2)$$

The natural question of the second-order behaviour of  $L_n$  was left largely open; similar questions may be posed about  $A_n$ .

In [23] a martingale-difference analysis was used to show that

$$\text{if } \mu \neq 0: \quad \lim_{n \rightarrow \infty} n^{-1} \mathbb{V} \text{ar } L_n = 4\sigma_\mu^2, \quad (3)$$

where we introduce the decomposition  $\sigma^2 = \sigma_\mu^2 + \sigma_{\mu^\perp}^2$  with

$$\sigma_\mu^2 := \mathbb{E} \left[ ((Z - \mu) \cdot \hat{\mu})^2 \right] = \mathbb{E}[(Z \cdot \hat{\mu})^2] - \|\mu\|^2 \in \mathbb{R}_+.$$

Here and elsewhere, ‘ $\cdot$ ’ denotes the scalar product,  $\hat{\mu} := \|\mu\|^{-1}\mu$  for  $\mu \neq 0$ , and  $\mathbb{R}_+ := [0, \infty)$ . In [23], a central limit theorem to accompany (3) was also obtained: provided  $\sigma_\mu^2 > 0$ ,  $n^{-1/2}(L_n - \mathbb{E} L_n)$  converges in distribution to a normal random variable with mean 0 and variance  $4\sigma_\mu^2$ . If  $\Sigma$  is positive definite, then both  $\sigma_\mu^2$  and  $\sigma_{\mu^\perp}^2$  are strictly positive, but our results are still of interest when one or other of them is zero (the case where both are zero being entirely trivial).

The aims of the present paper are to provide second-order information for  $L_n$  in the case  $\mu = 0$ , and to study the area  $A_n$  for both the cases  $\mu = 0$  and  $\mu \neq 0$ . For example, we will show that

$$\begin{aligned} \text{if } \mu \neq 0: & \quad \lim_{n \rightarrow \infty} n^{-3} \mathbb{V} \text{ar } A_n = v_+ \|\mu\|^2 \sigma_{\mu^\perp}^2; \\ \text{if } \mu = 0: & \quad \lim_{n \rightarrow \infty} n^{-1} \mathbb{V} \text{ar } L_n = u_0(\Sigma), \quad \text{and} \quad \lim_{n \rightarrow \infty} n^{-2} \mathbb{V} \text{ar } A_n = v_0 \det \Sigma. \end{aligned} \quad (4)$$

The quantities  $v_0$  and  $v_+$  in (4) are finite and positive, as is  $u_0(\cdot)$  provided  $\sigma^2 \in (0, \infty)$ , and these quantities are in fact variances associated with convex hulls of Brownian scaling limits for the walk. These scaling limits provide the basis of the analysis in this paper; the methods are necessarily quite different from those in [23]. The result  $\lim_{n \rightarrow \infty} n^{-1} \mathbb{V} \text{ar } L_n > 0$  in the case  $\mu = 0$  answers a question raised by Snyder and Steele [20, §5]. For the constants  $u_0(I)$  ( $I$  being the identity matrix),  $v_0$ , and  $v_+$ , Table 1 gives numerical evaluations of rigorous bounds that we prove in Proposition 3.7, plus estimates from simulations. See also Section 4 for an explicit integral expression for  $u_0(I)$ .

Table 2

Results originate from: <sup>a</sup> [21]; <sup>b</sup> [20]; <sup>c</sup> [23]; <sup>d</sup> [1] (in part); the rest are new. The limit laws exclude degenerate cases when associated variances vanish.

		Limit exists for $\mathbb{E}$	Limit exists for $\mathbb{V}\text{ar}$	Limit law
$\mu = 0$	$L_n$	$n^{-1/2}\mathbb{E}L_n^a$	$n^{-1}\mathbb{V}\text{ar}L_n$	Non-Gaussian
	$A_n$	$n^{-1}\mathbb{E}A_n^d$	$n^{-2}\mathbb{V}\text{ar}A_n$	Non-Gaussian
$\mu \neq 0$	$L_n$	$n^{-1}\mathbb{E}L_n^{a,b}$	$n^{-1}\mathbb{V}\text{ar}L_n^c$	Gaussian <sup>c</sup>
	$A_n$	$n^{-3/2}\mathbb{E}A_n$	$n^{-3}\mathbb{V}\text{ar}A_n$	Non-Gaussian

Furthermore, we show below that distributional limits accompanying the three variance asymptotics in (4) are *non-Gaussian*, excluding trivial cases, by contrast to the central limit theorem accompanying (3) from [23]. Also notable is the comparison between the variance asymptotics for  $\mu \neq 0$  in (3) and (4): each of the components  $\sigma_\mu^2$  and  $\sigma_{\mu_\perp}^2$  of  $\sigma^2$  contributes to exactly one of the asymptotics for  $\mathbb{V}\text{ar}L_n$  and  $\mathbb{V}\text{ar}A_n$ . Other results that we present below include asymptotics for expectations.

*Examples.* Here are some examples to illustrate a range of asymptotic behaviours exhibited by some simple models. We summarize what now is known in general in Table 2.

- Suppose that  $Z$  takes Cartesian vector values  $(1, 1)$ ,  $(-1, -1)$ ,  $(-1, 1)$  and  $(1, -1)$ , each with probability  $1/4$ . Then  $S_n$  is symmetric simple random walk on  $\mathbb{Z}^2$  with  $\mu = (0, 0)$  and  $\sigma^2 = 2$ . We show below that  $n^{-1/2}\mathbb{E}L_n \rightarrow \sqrt{8\pi}$  (see also [21]) and  $n^{-1}\mathbb{V}\text{ar}L_n \rightarrow u_0(I) \in (0, \infty)$ , while  $n^{-1}\mathbb{E}A_n \rightarrow \frac{\pi}{2}$  (see also [1]) and  $n^{-2}\mathbb{V}\text{ar}A_n \rightarrow v_0 \in (0, \infty)$ .
- Suppose  $Z$  takes values  $(1, 1)$  and  $(1, -1)$ , each with probability  $1/2$ . Then  $S_n$  can be viewed as the space–time diagram of *one-dimensional* simple symmetric random walk. Here  $\mu = (1, 0)$ ,  $\sigma_\mu^2 = 0$ , and  $\sigma_{\mu_\perp}^2 = 1$ . It is known that  $n^{-1}\mathbb{E}L_n \rightarrow 2$  [21,20] and  $\mathbb{V}\text{ar}L_n = o(n)$  [23]; we show below that  $n^{-3/2}\mathbb{E}A_n \rightarrow \frac{1}{3}\sqrt{2\pi}$  and  $n^{-3}\mathbb{V}\text{ar}A_n \rightarrow v_+ \in (0, \infty)$ .
- Suppose  $Z$  takes values  $(2, 0)$  and  $(0, 0)$ , each with probability  $1/2$ . Now  $\mu = (1, 0)$ ,  $\sigma_\mu^2 = 1$ , and  $\sigma_{\mu_\perp}^2 = 0$ . This time  $n^{-1}\mathbb{E}L_n \rightarrow 2$  [21,20] and  $n^{-1}\mathbb{V}\text{ar}L_n \rightarrow 4$  [23]; trivially  $A_n = 0$  a.s.

The outline of the rest of the paper is as follows. In Section 2 we describe our scaling limit approach, and carry it through after presenting the necessary preliminaries; the main results of this section, Theorems 2.5 and 2.7, give weak convergence statements for convex hulls of random walks in the case of zero and non-zero drift, respectively. Armed with these weak convergence results, we present asymptotics for expectations and variances of the quantities  $L_n$  and  $A_n$  in Section 3; the arguments in this section rely in part on the scaling limit apparatus, and in part on direct random walk computations. This section concludes with upper and lower bounds for the limiting variances. Section 4 collects some final remarks and open questions. Finally, the Appendix collects some auxiliary results on random walks that we use.

## 2. Scaling limits for convex hulls

### 2.1. Overview

We describe the general idea of our approach. Recall that  $S_n = \sum_{k=1}^n Z_k$  is the location of our random walk in  $\mathbb{R}^2$  after  $n$  steps. Write  $\mathcal{S}_n := \{S_0, S_1, \dots, S_n\}$ . Our strategy to study properties of

the random convex set hull  $\mathcal{S}_n$  (such as  $L_n$  or  $A_n$ ) is to seek a weak limit for a suitable scaling of hull  $\mathcal{S}_n$ , which we must hope to be the convex hull of some scaling limit representing the walk  $\mathcal{S}_n$ .

In the case of zero drift ( $\mu = 0$ ) a candidate scaling limit for the walk is readily identified in terms of planar Brownian motion. For the case  $\mu \neq 0$ , the ‘usual’ approach of centering and then scaling the walk (to again obtain planar Brownian motion) is not useful in our context, as this transformation does not act on the convex hull in any sensible way. A better idea is to scale space differently in the direction of  $\mu$  and in the orthogonal direction.

In other words, in either case we consider  $\phi_n(\mathcal{S}_n)$  for some *affine* continuous scaling function  $\phi_n : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . The convex hull is preserved under affine transformations, so

$$\phi_n(\text{hull } \mathcal{S}_n) = \text{hull } \phi_n(\mathcal{S}_n),$$

the convex hull of a random set which will have a weak limit. We will then be able to deduce scaling limits for quantities  $L_n$  and  $A_n$  provided, first, that we work in suitable spaces on which our functionals of interest enjoy continuity, so that we can appeal to the continuous mapping theorem for weak limits, and, second, that  $\phi_n$  acts on length and area by simple scaling. The usual  $n^{-1/2}$  scaling when  $\mu = 0$  is fine; for  $\mu \neq 0$  we scale space in one coordinate by  $n^{-1}$  and in the other by  $n^{-1/2}$ , which acts nicely on area, but *not* length. Thus these methods work in exactly the three cases corresponding to (4).

In view of the scaling limits that we expect, it is natural to work not with point sets like  $\mathcal{S}_n$ , but with continuous *paths*; instead of  $\mathcal{S}_n$  we consider the interpolating path constructed as follows. For each  $n \in \mathbb{N}$  and all  $t \in [0, 1]$ , define

$$X_n(t) := S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor)(S_{\lfloor nt \rfloor + 1} - S_{\lfloor nt \rfloor}) = S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor)Z_{\lfloor nt \rfloor + 1}.$$

Note that  $X_n(0) = S_0 = 0$  and  $X_n(1) = S_n$ . Given  $n$ , we are interested in the convex hull of the image in  $\mathbb{R}^2$  of the interval  $[0, 1]$  under the continuous function  $X_n$ . Our scaling limits will be of the same form.

## 2.2. Paths, hulls, and hulls of paths

We introduce the setting in which we will describe our scaling limit results. At this point, it is no extra difficulty to work in  $\mathbb{R}^d$  for general  $d \geq 2$ . Let  $\rho(x, y) = \|x - y\|$  denote the Euclidean distance between  $x$  and  $y$  in  $\mathbb{R}^d$ . For  $T > 0$ , let  $\mathcal{C}([0, T]; \mathbb{R}^d)$  denote the class of continuous functions from  $[0, T]$  to  $\mathbb{R}^d$ . Endow  $\mathcal{C}([0, T]; \mathbb{R}^d)$  with the supremum metric

$$\rho_\infty(f, g) := \sup_{t \in [0, T]} \rho(f(t), g(t)), \quad \text{for } f, g \in \mathcal{C}([0, T]; \mathbb{R}^d).$$

Let  $\mathcal{C}^0([0, T]; \mathbb{R}^d)$  denote those functions in  $\mathcal{C}([0, T]; \mathbb{R}^d)$  that map 0 to the origin in  $\mathbb{R}^d$ .

Usually, we work with  $T = 1$ , in which case we write simply

$$\mathcal{C}_d := \mathcal{C}([0, 1]; \mathbb{R}^d), \quad \text{and} \quad \mathcal{C}_d^0 := \{f \in \mathcal{C}_d : f(0) = 0\}.$$

For example,  $X_n \in \mathcal{C}_d^0$  for each  $n$ . For  $f \in \mathcal{C}([0, T]; \mathbb{R}^d)$  and  $t \in [0, T]$ , define  $f[0, t] := \{f(s) : s \in [0, t]\}$ , the image of  $[0, t]$  under  $f$ . Since  $[0, t]$  is compact and  $f$  is continuous, the *interval image*  $f[0, t]$  is compact. We view elements  $f \in \mathcal{C}([0, T]; \mathbb{R}^d)$  as *paths* indexed by time  $[0, T]$ , so that  $f[0, t]$  is the section of the path up to time  $t \in [0, T]$ .

We need some notation and concepts from convex and integral geometry: we found [9,19,21] to be very useful. For a set  $A \subseteq \mathbb{R}^d$ , write  $\partial A$  for its boundary and  $\text{int}(A) := A \setminus \partial A$  for its interior.

For  $A \subseteq \mathbb{R}^d$  and a point  $x \in \mathbb{R}^d$ , set  $\rho(x, A) := \inf_{y \in A} \rho(x, y)$ , with the usual convention that  $\inf \emptyset = +\infty$ . Write  $\mathbb{S}_{d-1} := \{e \in \mathbb{R}^d : \|e\| = 1\}$  for the unit sphere in  $\mathbb{R}^d$ .

Let  $\mathcal{K}_d$  denote the collection of convex compact sets in  $\mathbb{R}^d$ , and  $\mathcal{K}_d^0 := \{A \in \mathcal{K}_d : 0 \in A\}$  those that contain the origin. Given  $A \in \mathcal{K}_d$ , for  $r \geq 0$  set

$$\pi_r(A) := \{x \in \mathbb{R}^d : \rho(x, A) \leq r\},$$

the *parallel body* of  $A$  at distance  $r$ . The *support function* of  $A \in \mathcal{K}_d^0$  is  $h_A$  defined by

$$h_A(x) := \sup_{y \in A} (x \cdot y), \quad x \in \mathbb{R}^d.$$

Note that  $h_A : \mathbb{R}^d \rightarrow \mathbb{R}_+$  determines  $A$  via  $A = \{x : x \cdot e \leq h_A(e) \text{ for all } e \in \mathbb{S}_{d-1}\}$ , and that, for  $A, B \in \mathcal{K}_d^0$ , we have  $A \subseteq B$  if and only if  $h_A(e) \leq h_B(e)$  for all  $e \in \mathbb{S}_{d-1}$ ; see [9, p. 56]. The Hausdorff metric on  $\mathcal{K}_d^0$  is defined for  $A, B \in \mathcal{K}_d^0$  by

$$\rho_H(A, B) := \max \left\{ \sup_{x \in B} \rho(x, A), \sup_{y \in A} \rho(y, B) \right\}.$$

Two equivalent descriptions of  $\rho_H$  (see e.g. Proposition 6.3 of [9]) are

$$\rho_H(A, B) = \inf \{r \geq 0 : A \subseteq \pi_r(B) \text{ and } B \subseteq \pi_r(A)\}; \quad \text{and} \quad (5)$$

$$\rho_H(A, B) = \sup_{e \in \mathbb{S}_{d-1}} |h_A(e) - h_B(e)|. \quad (6)$$

For the rest of this section we study some basic properties of the map from a continuous path to its convex hull. Let  $f \in \mathcal{C}([0, T], \mathbb{R}^d)$ . For any  $t \in [0, T]$ ,  $f[0, t]$  is compact, and hence Carathéodory's theorem for convex hulls (see Corollary 3.1 of [9, p. 44]) shows that  $\text{hull } f[0, t]$  is also compact. So  $\text{hull } f[0, t] \in \mathcal{K}_d$  is convex, bounded, and closed; in particular, it is a Borel set.

It mostly suffices to work with paths parametrized over  $[0, 1]$ . For  $f \in \mathcal{C}_d$ , define

$$H(f) := \text{hull } f[0, 1].$$

The next result shows that the function  $H : (\mathcal{C}_d^0, \rho_\infty) \rightarrow (\mathcal{K}_d^0, \rho_H)$  is continuous.

**Lemma 2.1.** *For any  $f, g \in \mathcal{C}_d^0$ , we have  $H(f), H(g) \in \mathcal{K}_d^0$  and*

$$\rho_H(H(f), H(g)) \leq \rho_\infty(f, g). \quad (7)$$

**Proof.** Let  $f, g \in \mathcal{C}_d^0$ . Then  $H(f)$  and  $H(g)$  are non-empty, as they contain  $f(0) = g(0) = 0$ . Consider  $x \in H(f)$ . Since the convex hull of a set is the set of all convex combinations of points of the set (see Lemma 3.1 of [9, p. 42]), there exist  $n \in \mathbb{N}$ , weights  $\lambda_1, \dots, \lambda_n \geq 0$  with  $\sum_{i=1}^n \lambda_i = 1$ , and  $t_1, \dots, t_n \in [0, 1]$  for which  $x = \sum_{i=1}^n \lambda_i f(t_i)$ . Then, taking  $y = \sum_{i=1}^n \lambda_i g(t_i)$ , we have that  $y \in H(g)$  and, by the triangle inequality,

$$\rho(x, y) \leq \sum_{i=1}^n \lambda_i \rho(f(t_i), g(t_i)) \leq \rho_\infty(f, g).$$

Thus, writing  $r = \rho_\infty(f, g)$ , every  $x \in H(f)$  has  $x \in \pi_r(H(g))$ , i.e.,  $H(f) \subseteq \pi_r(H(g))$ . The symmetric argument starting with  $x \in H(g)$  shows that  $H(g) \subseteq \pi_r(H(f))$  as well. Hence, by (5), we obtain (7).  $\square$

We end this section by showing that the map  $t \mapsto \text{hull } f[0, t]$  on  $[0, T]$  is continuous if  $f$  is continuous on  $[0, T]$ , so that the continuous trajectory  $t \mapsto f(t)$  is accompanied by a continuous ‘trajectory’ of convex hulls. This observation was made by El Bachir [4, pp. 16–17]; we take a different route based on the path-space result [Lemma 2.1](#). First we need a lemma.

**Lemma 2.2.** *Let  $T > 0$  and  $f \in \mathcal{C}([0, T]; \mathbb{R}^d)$ . Then the map defined for  $t \in [0, T]$  by  $t \mapsto g_t$ , where  $g_t : [0, 1] \rightarrow \mathbb{R}^d$  is given by  $g_t(s) = f(ts)$ ,  $s \in [0, 1]$ , is a continuous function from  $([0, T], \rho)$  to  $(\mathcal{C}_d, \rho_\infty)$ .*

**Proof.** First we fix  $t \in [0, T]$  and show that  $s \mapsto g_t(s)$  is continuous, so that  $g_t \in \mathcal{C}_d$  as claimed. Since  $f$  is continuous on the compact interval  $[0, T]$ , it is uniformly continuous, and admits a monotone modulus of continuity  $\mu_f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\rho(f(s_1), f(s_2)) \leq \mu_f(\rho(s_1, s_2))$  for all  $s_1, s_2 \in [0, T]$ , and  $\mu_f(r) \downarrow 0$  as  $r \downarrow 0$  (see e.g. [12, p. 57]). Hence

$$\rho(g_t(s_1), g_t(s_2)) = \rho(f(ts_1), f(ts_2)) \leq \mu_f(\rho(ts_1, ts_2)) = \mu_f(t\rho(s_1, s_2)),$$

which tends to 0 as  $\rho(s_1, s_2) \rightarrow 0$ . Hence  $g_t \in \mathcal{C}_d$ .

It remains to show that  $t \mapsto g_t$  is continuous. But on  $\mathcal{C}_d$ ,

$$\begin{aligned} \rho_\infty(g_{t_1}, g_{t_2}) &= \sup_{s \in [0, 1]} \rho(f(t_1s), f(t_2s)) \\ &\leq \sup_{s \in [0, 1]} \mu_f(\rho(t_1s, t_2s)) \\ &= \mu_f(\rho(t_1, t_2)), \end{aligned}$$

which tends to 0 as  $\rho(t_1, t_2) \rightarrow 0$ , again using the uniform continuity of  $f$ .  $\square$

Here is the path continuity result for convex hulls of continuous paths; cf [4, pp. 16–17].

**Proposition 2.3.** *Let  $T > 0$  and  $f \in \mathcal{C}^0([0, T]; \mathbb{R}^d)$ . Then the map defined for  $t \in [0, T]$  by  $t \mapsto \text{hull } f[0, t]$  is a continuous function from  $([0, T], \rho)$  to  $(\mathcal{K}_d^0, \rho_H)$ .*

**Proof.** By [Lemma 2.2](#),  $t \mapsto g_t$  is continuous, where  $g_t(s) = f(ts)$ ,  $s \in [0, 1]$ . Note that, since  $f(0) = 0$ ,  $g_t \in \mathcal{C}_d^0$ . But the sets  $f[0, t]$  and  $g_t[0, 1]$  coincide, so  $\text{hull } f[0, t] = H(g_t) \in \mathcal{K}_d^0$ , and, by [Lemma 2.1](#),  $g_t \mapsto H(g_t)$  is continuous. Thus  $t \mapsto H(g_t)$  is the composition of two continuous functions, hence itself a continuous function.  $\square$

### 2.3. Functionals of planar convex hulls

Now, and for the rest of the paper, we return to  $d = 2$  to address our main questions of interest; parts of what follows carry over to general  $d \geq 2$ , but we do not pursue that generality here. We consider functionals  $\mathcal{A} : \mathcal{K}_2 \rightarrow \mathbb{R}_+$  and  $\mathcal{L} : \mathcal{K}_2 \rightarrow \mathbb{R}_+$  given by the area and the perimeter length of convex compact sets in the plane. Formally, we define  $\mathcal{A}$  as Lebesgue measure on  $\mathbb{R}^2$ , and then

$$\mathcal{L}(A) := \lim_{r \downarrow 0} \left( \frac{\mathcal{A}(\pi_r(A)) - \mathcal{A}(A)}{r} \right), \quad \text{for } A \in \mathcal{K}_2. \quad (8)$$

The limit in (8) exists by the *Steiner formula* of integral geometry (see e.g. [21]), which expresses  $\mathcal{A}(\pi_r(A))$  as a quadratic polynomial in  $r$  whose coefficients are given in terms of the *intrinsic volumes* of  $A$ :

$$\mathcal{A}(\pi_r(A)) = \mathcal{A}(A) + r\mathcal{L}(A) + \pi r^2 \mathbf{1}\{A \neq \emptyset\}. \quad (9)$$

In particular, with  $\mathcal{H}_d$  denoting  $d$ -dimensional Hausdorff measure,

$$\mathcal{L}(A) = \begin{cases} \mathcal{H}_1(\partial A) & \text{if } \text{int}(A) \neq \emptyset, \\ 2\mathcal{H}_1(\partial A) & \text{if } \text{int}(A) = \emptyset. \end{cases}$$

For  $A \in \mathcal{K}_2^0$ , Cauchy's formula states

$$\mathcal{L}(A) = \int_{\mathbb{S}_1} h_A(e) de.$$

It follows from Cauchy's formula that  $\mathcal{L}$  is increasing in the sense that if  $A, B \in \mathcal{K}_2^0$  satisfy  $A \subseteq B$ , then  $\mathcal{L}(A) \leq \mathcal{L}(B)$ ; clearly the functional  $\mathcal{A}$  is also increasing. The next result shows that the functions  $\mathcal{L}$  and  $\mathcal{A}$  are both continuous from  $(\mathcal{K}_2^0, \rho_H)$  to  $(\mathbb{R}_+, \rho)$ .

**Lemma 2.4.** *Suppose that  $A, B \in \mathcal{K}_2^0$ . Then*

$$\rho(\mathcal{L}(A), \mathcal{L}(B)) \leq 2\pi \rho_H(A, B); \quad (10)$$

$$\rho(\mathcal{A}(A), \mathcal{A}(B)) \leq \pi \rho_H(A, B)^2 + (\mathcal{L}(A) \vee \mathcal{L}(B)) \rho_H(A, B). \quad (11)$$

**Proof.** First consider  $\mathcal{L}$ . By Cauchy's formula and the triangle inequality,

$$|\mathcal{L}(A) - \mathcal{L}(B)| = \left| \int_{\mathbb{S}_1} (h_A(e) - h_B(e)) de \right| \leq 2\pi \sup_{e \in \mathbb{S}_1} |h_A(e) - h_B(e)|,$$

which with (6) gives (10).

Now consider  $\mathcal{A}$ . Set  $r = \rho_H(A, B)$ . Then, by (5),  $A \subseteq \pi_r(B)$ . Hence

$$\mathcal{A}(A) \leq \mathcal{A}(\pi_r(B)) \leq \mathcal{A}(B) + r\mathcal{L}(B) + \pi r^2,$$

by (9). With the symmetric argument starting from  $B \subseteq \pi_r(A)$ , we get (11).  $\square$

## 2.4. Brownian convex hulls as scaling limits

The two different scalings outlined in Section 2.1, for the cases  $\mu = 0$  and  $\mu \neq 0$ , lead to different scaling limits for the random walk. Both are associated with Brownian motion.

In the case  $\mu = 0$ , the scaling limit is the usual planar Brownian motion, at least when  $\Sigma = I$ , the identity matrix. Let  $b := (b(s))_{s \in [0,1]}$  denote standard Brownian motion in  $\mathbb{R}^2$ , started at  $b(0) = 0$ . For convenience we may assume  $b \in \mathcal{C}_2^0$  (we can work on a probability space for which continuity holds for all sample points, rather than merely almost all). For  $t \in [0, 1]$ , let  $h_t := \text{hull } b[0, t] \in \mathcal{K}_2^0$  denote the convex hull of the Brownian path up to time  $t$ . By Proposition 2.3,  $t \mapsto h_t$  is continuous. Much is known about the properties of  $h_t$ : see e.g. [2,4,6,13]. We also set

$$\ell_t := \mathcal{L}(h_t), \quad \text{and} \quad a_t := \mathcal{A}(h_t),$$

the perimeter length and area of the standard Brownian convex hull. By Lemma 2.4, the processes  $t \mapsto \ell_t$  and  $t \mapsto a_t$  have continuous and non-decreasing sample paths.

We also need to work with the case of general covariances  $\Sigma$ ; to do so we introduce more notation and recall some facts about multivariate Gaussian random vectors. For definiteness, we view vectors as Cartesian column vectors when required. Since  $\Sigma$  is positive semidefinite and symmetric, there is a (unique) positive semidefinite symmetric matrix square-root  $\Sigma^{1/2}$  for which



$\Sigma = (\Sigma^{1/2})^2$ . The map  $x \mapsto \Sigma^{1/2}x$  associated with  $\Sigma^{1/2}$  is a linear transformation on  $\mathbb{R}^2$  with Jacobian  $\det \Sigma^{1/2} = \sqrt{\det \Sigma}$ ; hence  $\mathcal{A}(\Sigma^{1/2}A) = \mathcal{A}(A)\sqrt{\det \Sigma}$  for any measurable  $A \subseteq \mathbb{R}^2$ .

If  $W \sim \mathcal{N}(0, I)$ , then  $\Sigma^{1/2}W \sim \mathcal{N}(0, \Sigma)$ , a bivariate normal distribution with mean 0 and covariance  $\Sigma$ ; the notation permits  $\Sigma = 0$ , in which case  $\mathcal{N}(0, 0)$  stands for the degenerate normal distribution with point mass at 0. Similarly, given  $b$  a standard Brownian motion on  $\mathbb{R}^2$ , the diffusion  $\Sigma^{1/2}b$  is *correlated* planar Brownian motion with covariance matrix  $\Sigma$ . We write ‘ $\Rightarrow$ ’ to indicate weak convergence.

**Theorem 2.5.** *Suppose that  $\mu = 0$ . Then, as  $n \rightarrow \infty$ ,*

$$n^{-1/2} \text{hull}\{S_0, S_1, \dots, S_n\} \Rightarrow \Sigma^{1/2}h_1,$$

*in the sense of weak convergence on  $(\mathcal{K}_2^0, \rho_H)$ .*

**Proof.** Donsker’s theorem implies that  $n^{-1/2}X_n \Rightarrow \Sigma^{1/2}b$  on  $(\mathcal{C}_2^0, \rho_\infty)$ . Now, the point set  $X_n[0, 1]$  is the union of the line segments  $\{S_k + \theta(S_{k+1} - S_k) : \theta \in [0, 1]\}$  over  $k = 0, 1, \dots, n-1$ . Since the convex hull is preserved under affine transformations,

$$H(n^{-1/2}X_n) = n^{-1/2}H(X_n) = n^{-1/2} \text{hull}\{S_0, S_1, \dots, S_n\}.$$

By Lemma 2.1,  $H$  is continuous, and so the continuous mapping theorem (see e.g. [12, p. 76]) implies that  $n^{-1/2} \text{hull}\{S_0, S_1, \dots, S_n\} \Rightarrow H(\Sigma^{1/2}b)$  on  $(\mathcal{K}_2^0, \rho_H)$ . Finally, invariance of the convex hull under affine transformations shows  $H(\Sigma^{1/2}b) = \Sigma^{1/2}H(b) = \Sigma^{1/2}h_1$ .  $\square$

Theorem 2.5 together with the continuous mapping theorem and Lemma 2.4 implies the following distributional limit results in the case  $\mu = 0$ . Here and subsequently ‘ $\xrightarrow{d}$ ’ denotes convergence in distribution for  $\mathbb{R}$ -valued random variables.

**Corollary 2.6.** *Suppose that  $\mu = 0$ . Then, as  $n \rightarrow \infty$ ,*

$$n^{-1/2}L_n \xrightarrow{d} \mathcal{L}(\Sigma^{1/2}h_1), \quad \text{and} \quad n^{-1}A_n \xrightarrow{d} \mathcal{A}(\Sigma^{1/2}h_1) = a_1\sqrt{\det \Sigma}.$$

**Remark.** If a real-valued random variable  $X$  is Gaussian and non-degenerate (i.e.,  $\text{Var } X > 0$ ), then  $\text{ess inf } X = -\infty$  and  $\text{ess sup } X = +\infty$ . The distributional limits for  $n^{-1/2}L_n$  and  $n^{-1}A_n$  in Corollary 2.6 are supported on  $\mathbb{R}_+$  and, as we will show in Proposition 3.7, are non-degenerate if  $\Sigma$  is positive definite; hence they are *non-Gaussian* excluding trivial cases.

In the case  $\mu \neq 0$ , the scaling limit can be viewed as a space–time trajectory of one-dimensional Brownian motion. Let  $w := (w(s))_{s \in [0,1]}$  denote standard Brownian motion in  $\mathbb{R}$ , started at  $w(0) = 0$ ; similarly to above, we may take  $w \in \mathcal{C}_1^0$ . Define  $\tilde{b} \in \mathcal{C}_2^0$  in Cartesian coordinates via

$$\tilde{b}(s) = (s, w(s)), \quad \text{for } s \in [0, 1];$$

thus  $\tilde{b}[0, 1]$  is the space–time diagram of one-dimensional Brownian motion run for unit time. For  $t \in [0, 1]$ , let  $\tilde{h}_t := \text{hull } \tilde{b}[0, t] \in \mathcal{K}_2^0$ , and define  $\tilde{a}_t := \mathcal{A}(\tilde{h}_t)$ . (Closely related to  $\tilde{h}_t$  is the greatest *convex minorant* of  $w$  over  $[0, t]$ , which is of interest in its own right, see e.g. [17] and references therein.)

Suppose  $\mu \neq 0$  and  $\sigma_{\mu_\perp}^2 \in (0, \infty)$ . Given  $\mu \in \mathbb{R}^2 \setminus \{0\}$ , let  $\hat{\mu}_\perp$  be the unit vector perpendicular to  $\mu$  obtained by rotating  $\hat{\mu}$  by  $\pi/2$  anticlockwise. For  $n \in \mathbb{N}$ , define  $\psi_n^\mu : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by the image

of  $x \in \mathbb{R}^2$  in Cartesian components:

$$\psi_n^\mu(x) = \left( \frac{x \cdot \hat{\mu}}{n\|\mu\|}, \frac{x \cdot \hat{\mu}_\perp}{\sqrt{n\sigma_{\mu_\perp}^2}} \right).$$

In words,  $\psi_n^\mu$  rotates  $\mathbb{R}^2$ , mapping  $\hat{\mu}$  to the unit vector in the horizontal direction, and then scales space with a horizontal shrinking factor  $\|\mu\|n$  and a vertical factor  $\sqrt{n\sigma_{\mu_\perp}^2}$ ; see Fig. 2 for an illustration.

**Theorem 2.7.** Suppose that  $\mu \neq 0$ , and  $\sigma_{\mu_\perp}^2 > 0$ . Then, as  $n \rightarrow \infty$ ,

$$\psi_n^\mu(\text{hull}\{S_0, S_1, \dots, S_n\}) \Rightarrow \tilde{h}_1,$$

in the sense of weak convergence on  $(\mathcal{K}_2^0, \rho_H)$ .

**Proof.** Observe that  $\hat{\mu} \cdot S_n$  is a random walk on  $\mathbb{R}$  with one-step mean drift  $\hat{\mu} \cdot \mu = \|\mu\| \in (0, \infty)$ , while  $\hat{\mu}_\perp \cdot S_n$  is a walk with mean drift  $\hat{\mu}_\perp \cdot \mu = 0$  and increment variance

$$\begin{aligned} \mathbb{E}[(\hat{\mu}_\perp \cdot Z)^2] &= \mathbb{E}[(\hat{\mu}_\perp \cdot (Z - \mu))^2] = \mathbb{E}[\|Z - \mu\|^2] - \mathbb{E}[(\hat{\mu} \cdot (Z - \mu))^2] \\ &= \sigma^2 - \sigma_\mu^2 = \sigma_{\mu_\perp}^2. \end{aligned}$$

According to the strong law of large numbers, for any  $\varepsilon > 0$  there exists  $N_\varepsilon \in \mathbb{N}$  a.s. such that  $|m^{-1}\hat{\mu} \cdot S_m - \|\mu\|| < \varepsilon$  for all  $m \geq N_\varepsilon$ . Now we have that

$$\begin{aligned} \sup_{N_\varepsilon/n \leq t \leq 1} \left| \frac{\hat{\mu} \cdot S_{\lfloor nt \rfloor}}{n} - t\|\mu\| \right| &\leq \sup_{N_\varepsilon/n \leq t \leq 1} \left( \frac{\lfloor nt \rfloor}{n} \right) \left| \frac{\hat{\mu} \cdot S_{\lfloor nt \rfloor}}{\lfloor nt \rfloor} - \|\mu\| \right| \\ &\quad + \|\mu\| \sup_{0 \leq t \leq 1} \left| \frac{\lfloor nt \rfloor}{n} - t \right| \\ &\leq \sup_{N_\varepsilon/n \leq t \leq 1} \left| \frac{\hat{\mu} \cdot S_{\lfloor nt \rfloor}}{\lfloor nt \rfloor} - \|\mu\| \right| + \frac{\|\mu\|}{n} \leq \varepsilon + \frac{\|\mu\|}{n}. \end{aligned}$$

On the other hand,

$$\sup_{0 \leq t \leq N_\varepsilon/n} \left| \frac{\hat{\mu} \cdot S_{\lfloor nt \rfloor}}{n} - t\|\mu\| \right| \leq \frac{1}{n} \max\{\hat{\mu} \cdot S_0, \dots, \hat{\mu} \cdot S_{N_\varepsilon}\} + \frac{N_\varepsilon\|\mu\|}{n} \rightarrow 0, \text{ a.s.,}$$

since  $N_\varepsilon < \infty$  a.s. Combining these last two displays and using the fact that  $\varepsilon > 0$  was arbitrary, we see that  $\sup_{0 \leq t \leq 1} |n^{-1}\hat{\mu} \cdot S_{\lfloor nt \rfloor} - t\|\mu\|| \rightarrow 0$ , a.s. (the functional version of the strong law). Similarly,  $\sup_{0 \leq t \leq 1} |n^{-1}\hat{\mu} \cdot S_{\lfloor nt \rfloor+1} - t\|\mu\|| \rightarrow 0$ , a.s. as well. Since  $X_n(t)$  interpolates  $S_{\lfloor nt \rfloor}$  and  $S_{\lfloor nt \rfloor+1}$ , it follows that  $\sup_{0 \leq t \leq 1} |n^{-1}\hat{\mu} \cdot X_n(t) - t\|\mu\|| \rightarrow 0$ , a.s. In other words,  $(n\|\mu\|)^{-1}X_n \cdot \hat{\mu}$  converges a.s. to the identity function  $t \mapsto t$  on  $[0, 1]$ .

For the other component, Donsker's theorem gives  $(n\sigma_{\mu_\perp}^2)^{-1/2}X_n \cdot \hat{\mu}_\perp \Rightarrow w$  on  $(\mathcal{C}_1^0, \rho_\infty)$ . It follows that, as  $n \rightarrow \infty$ ,  $\psi_n^\mu(X_n) \Rightarrow \tilde{b}$ , on  $(\mathcal{C}_2^0, \rho_\infty)$ . Hence by Lemma 2.1 and since  $\psi_n^\mu$  acts as an affine transformation on  $\mathbb{R}^2$ ,

$$\psi_n^\mu(H(X_n)) = H(\psi_n^\mu(X_n)) \Rightarrow H(\tilde{b}),$$

on  $(\mathcal{K}_2^0, \rho_H)$ , and the result follows.  $\square$

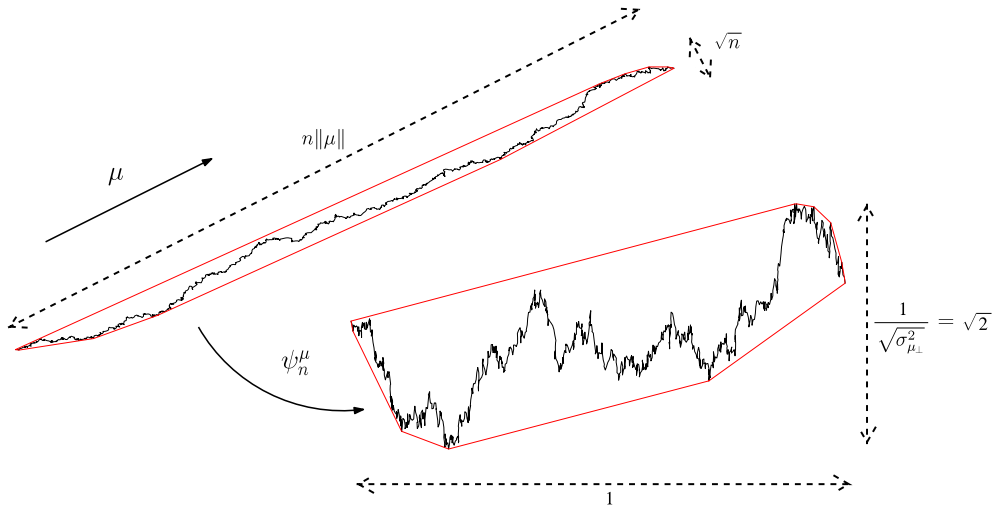


Fig. 2. Simulated path of  $n = 1000$  steps a random walk with drift  $\mu = (\frac{1}{2}, \frac{1}{4})$  and its convex hull (top left) and (not to the same scale) the image under  $\psi_n^{\mu}$  (bottom right).

**Theorem 2.7** with the continuous mapping theorem, **Lemma 2.4**, and the fact that  $\mathcal{A}(\psi_n^{\mu}(A)) = n^{-3/2}\|\mu\|^{-1}(\sigma_{\mu_{\perp}}^2)^{-1/2}\mathcal{A}(A)$  for measurable  $A \subseteq \mathbb{R}^2$ , implies the following distributional limit for  $A_n$  in the case  $\mu \neq 0$ .

**Corollary 2.8.** Suppose that  $\mu \neq 0$ , and  $\sigma_{\mu_{\perp}}^2 > 0$ . Then

$$n^{-3/2}A_n \xrightarrow{d} \|\mu\|(\sigma_{\mu_{\perp}}^2)^{1/2}\tilde{a}_1, \quad \text{as } n \rightarrow \infty.$$

**Remarks.** (i) Only the  $\sigma_{\mu_{\perp}}^2 > 0$  case is non-trivial, since  $\sigma_{\mu_{\perp}}^2 = 0$  if and only if  $Z$  is parallel to  $\pm\mu$  a.s., in which case all the points  $S_0, \dots, S_n$  are collinear and  $A_n = 0$  a.s. for all  $n$ .

(ii) The limit in **Corollary 2.8** is non-negative and non-degenerate (see **Proposition 3.7**) and hence non-Gaussian.

### 3. Expectation and variance asymptotics

#### 3.1. Expectation asymptotics

We start with asymptotics for  $\mathbb{E}L_n$  and  $\mathbb{E}A_n$  in the case  $\mu = 0$ . These results, **Propositions 3.1** and **3.3**, are in part already contained in [21] and [1] respectively; we give concise proofs here since several of the computations involved will be useful later. The first result, essentially given in [21, p. 508], is for  $L_n$ .

**Proposition 3.1.** Suppose that  $\mu = 0$ . Then, for  $Y \sim \mathcal{N}(0, \Sigma)$ ,

$$\lim_{n \rightarrow \infty} n^{-1/2}\mathbb{E}L_n = \mathbb{E}\mathcal{L}(\Sigma^{1/2}h_1) = 4\mathbb{E}\|Y\|.$$

Cauchy's formula applied to the line segment from 0 to  $Y$  with Fubini's theorem implies  $2\mathbb{E}\|Y\| = \int_{\mathbb{S}^1} \mathbb{E}[(Y \cdot e)^+] de$ . Here  $Y \cdot e = e^{\top}Y$  is univariate normal with mean 0 and variance

$e^\top \Sigma e = \|\Sigma^{1/2} e\|^2$ , so that  $\mathbb{E}[(Y \cdot e)^+]$  is  $\|\Sigma^{1/2} e\|$  times one half of the mean of the square-root of a  $\chi_1^2$  random variable. Hence  $\mathbb{E}\|Y\| = (8\pi)^{-1/2} \int_{\mathbb{S}_1} \|\Sigma^{1/2} e\| de$ , which in general may be expressed via a complete elliptic integral of the second kind in terms of the ratio of the eigenvalues of  $\Sigma$ . In the particular case  $\Sigma = I$ ,  $\mathbb{E}\|Y\| = \sqrt{\pi/2}$  so then Proposition 3.1 implies that

$$\lim_{n \rightarrow \infty} n^{-1/2} \mathbb{E} L_n = \sqrt{8\pi},$$

matching the formula  $\mathbb{E} \ell_1 = \sqrt{8\pi}$  of Letac and Takács [14,22]. We also note the bounds

$$\pi^{-1/2} \sqrt{\text{tr } \Sigma} \leq \mathbb{E}\|Y\| \leq \sqrt{\text{tr } \Sigma}; \quad (12)$$

the upper bound here is from Jensen's inequality and the fact that  $\mathbb{E}[\|Y\|^2] = \text{tr } \Sigma$ . The lower bound in (12) follows from the inequality

$$\mathbb{E}\|Y\| \geq \sup_{e \in \mathbb{S}_1} \mathbb{E}|Y \cdot e| = \sqrt{2/\pi} \sup_{e \in \mathbb{S}_1} (\text{Var}[Y \cdot e])^{1/2}$$

together with the fact that

$$\sup_{e \in \mathbb{S}_1} \text{Var}[Y \cdot e] = \sup_{e \in \mathbb{S}_1} \|\Sigma^{1/2} e\|^2 = \|\Sigma^{1/2}\|_{\text{op}}^2 = \|\Sigma\|_{\text{op}} = \lambda_\Sigma \geq \frac{1}{2} \text{tr } \Sigma,$$

where  $\|\cdot\|_{\text{op}}$  is the matrix operator norm and  $\lambda_\Sigma$  is the largest eigenvalue of  $\Sigma$ ; in statistical terminology,  $\lambda_\Sigma$  is the variance of the first principal component associated with  $Y$ .

In what follows, we make repeated use of the following fact (see e.g. [12, Lemma 4.11]): if random variables  $\zeta, \zeta_1, \zeta_2, \dots$  are such that  $\zeta_n \rightarrow \zeta$  in distribution and the  $\zeta_n$  are uniformly integrable, then  $\mathbb{E}\zeta_n \rightarrow \mathbb{E}\zeta$ .

**Proof of Proposition 3.1.** The finite point-set case of Cauchy's formula gives

$$L_n = \int_{\mathbb{S}_1} \max_{0 \leq k \leq n} (S_k \cdot e) de \leq 2\pi \max_{0 \leq k \leq n} \|S_k\|. \quad (13)$$

Then by Lemma A.1(ii) we have  $\sup_n \mathbb{E}[(n^{-1/2} L_n)^2] < \infty$ . Hence  $n^{-1/2} L_n$  is uniformly integrable, so that Corollary 2.6 yields  $\lim_{n \rightarrow \infty} n^{-1/2} \mathbb{E} L_n = \mathbb{E} \mathcal{L}(\Sigma^{1/2} h_1)$ .

It remains to show that  $\lim_{n \rightarrow \infty} n^{-1/2} \mathbb{E} L_n = 4\mathbb{E}\|Y\|$ . One can use Cauchy's formula to compute  $\mathbb{E} \mathcal{L}(\Sigma^{1/2} h_1)$ ; instead we give a direct random walk argument, following [21]. The central limit theorem for  $S_n$  implies that  $n^{-1/2} \|S_n\| \rightarrow \|Y\|$  in distribution. Under the given conditions,  $\mathbb{E}[\|S_{n+1}\|^2] = \mathbb{E}[\|S_n\|^2] + \mathbb{E}[\|Z\|^2]$ , so that  $\mathbb{E}[\|S_n\|^2] = O(n)$ . It follows that  $n^{-1/2} \|S_n\|$  is uniformly integrable, and hence  $\lim_{n \rightarrow \infty} n^{-1/2} \mathbb{E} \|S_n\| = \mathbb{E}\|Y\|$ . The result now follows from some standard analysis based on (1) and the fact that  $\lim_{n \rightarrow \infty} n^{-1/2} \sum_{k=1}^n k^{-1/2} = 2$ .

Now we move on to the area  $A_n$ . First we state some useful moments bounds.

**Lemma 3.2.** Let  $p \geq 1$ . Suppose that  $\mathbb{E}[\|Z\|^{2p}] < \infty$ .

- (i) We have  $\mathbb{E}[A_n^p] = O(n^{3p/2})$ .
- (ii) Moreover, if  $\mu = 0$  we have  $\mathbb{E}[A_n^p] = O(n^p)$ .

**Proof.** First we prove (ii). Since  $\text{hull}\{S_0, \dots, S_n\}$  is contained in the disk of radius  $\max_{0 \leq m \leq n} \|S_m\|$  and centre 0, we have  $A_n^p \leq \pi^p \max_{0 \leq m \leq n} \|S_m\|^{2p}$ . Lemma A.1(ii) then yields part (ii).

For part (i), it suffices to suppose  $\mu \neq 0$ . Then, bounding the convex hull by a rectangle,

$$\begin{aligned} A_n &\leq \left( \max_{0 \leq m \leq n} S_m \cdot \hat{\mu} - \min_{0 \leq m \leq n} S_m \cdot \hat{\mu} \right) \left( \max_{0 \leq m \leq n} S_m \cdot \hat{\mu}_\perp - \min_{0 \leq m \leq n} S_m \cdot \hat{\mu}_\perp \right) \\ &\leq 4 \left( \max_{0 \leq m \leq n} |S_m \cdot \hat{\mu}| \right) \left( \max_{0 \leq m \leq n} |S_m \cdot \hat{\mu}_\perp| \right). \end{aligned}$$

Hence, by the Cauchy–Schwarz inequality, we have

$$\mathbb{E}[A_n^p] \leq 4^p \left( \mathbb{E} \left[ \max_{0 \leq m \leq n} |S_m \cdot \hat{\mu}|^{2p} \right] \right)^{1/2} \left( \mathbb{E} \left[ \max_{0 \leq m \leq n} |S_m \cdot \hat{\mu}_\perp|^{2p} \right] \right)^{1/2}.$$

Now an application of [Lemma A.1](#)(i) and (iii) gives part (i).  $\square$

The asymptotics for  $\mathbb{E}A_n$  in the case  $\mu = 0$  are given in the following result, which is in part contained in [1, p. 325].

**Proposition 3.3.** *Suppose that  $\mu = 0$ . Then,*

$$\lim_{n \rightarrow \infty} n^{-1} \mathbb{E}A_n = \frac{\pi}{2} \sqrt{\det \Sigma}.$$

Given [Corollary 2.6](#), one may also deduce the limit result in [Proposition 3.3](#) from the formula  $\mathbb{E}a_1 = \frac{\pi}{2}$  of El Bachir [4, p. 66] with a uniform integrability argument; however, the naïve approach seems to require a slightly stronger moments assumption, such as  $(M_p)$  for some  $p > 2$  (cf [Lemma 3.2](#)). The proof of [Proposition 3.3](#) is based on an analogue for  $\mathbb{E}A_n$  of the Spitzer–Widom formula, due to Barndorff-Nielsen and Baxter [1]. To state the formula, let  $T(u, v)$  ( $u, v \in \mathbb{R}^2$ ) be the area of a triangle with sides of  $u, v$  and  $u + v$ . Note that for  $\alpha, \beta > 0$ ,  $T(\alpha u, \beta v) = \alpha\beta T(u, v)$ . The formula of [1] states

$$\mathbb{E}A_n = \sum_{k=2}^n \sum_{m=1}^{k-1} \frac{\mathbb{E}T(S_m, S_k - S_m)}{m(k-m)}. \quad (14)$$

**Proof of Proposition 3.3.** First we show that, under the given conditions,

$$\lim_{m \rightarrow \infty, k-m \rightarrow \infty} \frac{\mathbb{E}T(S_m, S_k - S_m)}{\sqrt{m(k-m)}} = \mathbb{E}T(Y_1, Y_2), \quad (15)$$

where  $Y_1$  and  $Y_2$  are independent  $\mathcal{N}(0, \Sigma)$  random vectors. Indeed, it follows from the central limit theorem in  $\mathbb{R}^2$  and the continuity of  $T$  that

$$\frac{T(S_m, S_k - S_m)}{\sqrt{m(k-m)}} = T\left(\frac{S_m}{\sqrt{m}}, \frac{S_k - S_m}{\sqrt{k-m}}\right) \xrightarrow{d} T(Y_1, Y_2),$$

as  $m \rightarrow \infty$  and  $k - m \rightarrow \infty$ . Moreover,  $T(u, v) \leq \|u\| \|v\|$  so

$$\begin{aligned} \mathbb{E} \left[ \left( \frac{T(S_m, S_k - S_m)}{\sqrt{m(k-m)}} \right)^2 \right] &\leq \frac{\mathbb{E}[\|S_m\|^2 \|S_k - S_m\|^2]}{m(k-m)} \\ &\leq \frac{\mathbb{E}[\|S_m\|^2]}{m} \cdot \frac{\mathbb{E}[\|S_k - S_m\|^2]}{k-m}, \end{aligned}$$

which is uniformly bounded for  $k \geq m + 1 \geq 0$ , by Lemma A.1. It follows that  $m^{-1/2}(k - m)^{-1/2}T(S_m, S_k - S_m)$  is uniformly integrable over  $(m, k)$  with  $m \geq 1$ ,  $k \geq m + 1$ , and the claim (15) follows.

With  $\Sigma = (\Sigma^{1/2})^2$ , we have that  $(Y_1, Y_2)$  is equal in distribution to  $(\Sigma^{1/2}W_1, \Sigma^{1/2}W_2)$  where  $W_1$  and  $W_2$  are independent  $\mathcal{N}(0, I)$  random vectors. Since  $\Sigma^{1/2}$  acts as a linear transformation on  $\mathbb{R}^2$  with Jacobian  $\sqrt{\det \Sigma}$ ,

$$\mathbb{E}T(Y_1, Y_2) = \mathbb{E}T(\Sigma^{1/2}W_1, \Sigma^{1/2}W_2) = \sqrt{\det \Sigma} \mathbb{E}T(W_1, W_2).$$

Here  $\mathbb{E}T(W_1, W_2) = \frac{1}{2} \mathbb{E}[\|W_1\| \|W_2\| \sin \Theta]$ , where the minimum angle  $\Theta$  between  $W_1$  and  $W_2$  is uniform on  $[0, \pi]$ , and  $(\|W_1\|, \|W_2\|, \Theta)$  are independent. Hence  $\mathbb{E}T(W_1, W_2) = \frac{1}{2}(\mathbb{E}\|W_1\|)^2(\mathbb{E} \sin \Theta) = \frac{1}{2}$ , using the fact that  $\mathbb{E} \sin \Theta = 2/\pi$  and  $\|W_1\|$  is the square-root of a  $\chi_2^2$  random variable, so  $\mathbb{E}\|W_1\| = \sqrt{\pi/2}$ .

Thus from (14), (15), and the computation  $\mathbb{E}T(Y_1, Y_2) = \frac{1}{2}\sqrt{\det \Sigma}$ , we have

$$\mathbb{E}A_n = \frac{1}{2}\sqrt{\det \Sigma} \sum_{k=2}^n \sum_{m=1}^{k-1} m^{-1/2}(k-m)^{-1/2} (1 + \varepsilon_{k,m}), \quad (16)$$

where, for any  $\varepsilon > 0$ , there exists  $m_0 \in \mathbb{N}$  such that  $|\varepsilon_{k,m}| \leq \varepsilon$  for all  $m \geq m_0$  and  $k - m \geq m_0$ . Moreover,

$$\lim_{k \rightarrow \infty} \sum_{m=1}^{k-1} m^{-1/2}(k-m)^{-1/2} = \int_0^1 y^{-1/2}(1-y)^{-1/2} dy = \pi, \quad (17)$$

so that the corresponding Cesàro limit also satisfies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=2}^n \sum_{m=1}^{k-1} m^{-1/2}(k-m)^{-1/2} = \pi.$$

With (16) it follows that, for any  $\varepsilon > 0$ ,

$$n^{-1} \mathbb{E}A_n \leq \frac{\pi}{2}(1 + \varepsilon)\sqrt{\det \Sigma} + O(n^{-1/2}),$$

which gives  $\limsup_{n \rightarrow \infty} n^{-1} \mathbb{E}A_n \leq \frac{\pi}{2}\sqrt{\det \Sigma}$ , and a similar argument gives the corresponding  $\liminf$  result.

Next we move on to the case  $\mu \neq 0$ . The following result on the asymptotics of  $\mathbb{E}A_n$  in this case is, as far as we are aware, new.

**Proposition 3.4.** Suppose that  $(M_p)$  holds for some  $p > 2$ ,  $\mu \neq 0$ , and  $\sigma_{\mu_\perp}^2 > 0$ . Then

$$\lim_{n \rightarrow \infty} n^{-3/2} \mathbb{E}A_n = \|\mu\|(\sigma_{\mu_\perp}^2)^{1/2} \mathbb{E}\tilde{a}_1 = \frac{1}{3} \|\mu\| \sqrt{2\pi \sigma_{\mu_\perp}^2}.$$

In particular,  $\mathbb{E}\tilde{a}_1 = \frac{1}{3}\sqrt{2\pi}$ .

**Proof.** Given  $\mathbb{E}[\|Z_1\|^p] < \infty$  for some  $p > 2$ , Lemma 3.2(i) shows that  $\mathbb{E}[A_n^{p/2}] = O(n^{3p/4})$ , so that  $\mathbb{E}[(n^{-3/2}A_n)^{p/2}]$  is uniformly bounded. Hence  $n^{-3/2}A_n$  is uniformly integrable, so Corollary 2.8 implies that

$$\lim_{n \rightarrow \infty} n^{-3/2} \mathbb{E}A_n = \|\mu\|(\sigma_{\mu_\perp}^2)^{1/2} \mathbb{E}\tilde{a}_1. \quad (18)$$

In light of (18), it remains to identify  $\mathbb{E}\tilde{a}_1 = \frac{1}{3}\sqrt{2\pi}$ . It does not seem straightforward to work directly with the Brownian limit; it turns out again to be simpler to work with a suitable random walk. We choose a walk that is particularly convenient for computations.

Let  $\xi \sim \mathcal{N}(0, 1)$  be a standard normal random variable, and take  $Z$  to be distributed as  $Z = (1, \xi)$  in Cartesian coordinates. Then  $S_n = (n, \sum_{k=1}^n \xi_k)$  is the space–time diagram of the symmetric random walk on  $\mathbb{R}$  generated by i.i.d. copies  $\xi_1, \xi_2, \dots$  of  $\xi$ .

For  $Z = (1, \xi)$ ,  $\mu = (1, 0)$  and  $\sigma^2 = \sigma_{\mu^\perp}^2 = \mathbb{E}[\xi^2] = 1$ . Thus by (18), to complete the proof of Proposition 3.4 it suffices to show that for this walk  $\lim_{n \rightarrow \infty} n^{-3/2} \mathbb{E}A_n = \frac{1}{3}\sqrt{2\pi}$ . If  $u, v \in \mathbb{R}^2$  have Cartesian components  $u = (u_1, u_2)$  and  $v = (v_1, v_2)$ , then we may write  $T(u, v) = \frac{1}{2}|u_1 v_2 - v_1 u_2|$ . Hence

$$T(S_m, S_k - S_m) = \frac{1}{2} \left| (k-m) \sum_{j=1}^m \xi_j - m \sum_{j=m+1}^k \xi_j \right|.$$

By properties of the normal distribution, the right-hand side of the last display has the same distribution as  $\frac{1}{2}|\xi|\sqrt{km(k-m)}|$ . Hence

$$\frac{\mathbb{E}T(S_m, S_k - S_m)}{\sqrt{m(k-m)}} = \frac{1}{2}\mathbb{E}|\xi|\sqrt{k} = \frac{1}{2}\sqrt{2k/\pi},$$

using the fact that  $|\xi|$  is distributed as the square-root of a  $\chi_1^2$  random variable, so  $\mathbb{E}|\xi| = \sqrt{2/\pi}$ . Hence, by (14), this random walk enjoys the exact formula

$$\mathbb{E}A_n = \frac{1}{\sqrt{2\pi}} \sum_{k=2}^n \sum_{m=1}^{k-1} \frac{\sqrt{k}}{\sqrt{m(k-m)}}.$$

Then from (17) we obtain  $\mathbb{E}A_n \sim \sqrt{\pi/2} \sum_{k=2}^n k^{1/2}$ , which gives the result.  $\square$

### 3.2. Variance asymptotics

We are now able to give formally the results quoted in (4), and to explain the constants that appear in the limits. Indeed, these are defined to be

$$u_0(\Sigma) := \mathbb{V}\text{ar } \mathcal{L}(\Sigma^{1/2}h_1), \quad v_0 := \mathbb{V}\text{ar } a_1, \quad v_+ := \mathbb{V}\text{ar } \tilde{a}_1. \quad (19)$$

**Proposition 3.5.** Suppose that  $(M_p)$  holds for some  $p > 2$ , and  $\mu = 0$ . Then

$$\lim_{n \rightarrow \infty} n^{-1} \mathbb{V}\text{ar } L_n = u_0(\Sigma).$$

If, in addition,  $(M_p)$  holds for some  $p > 4$ , then

$$\lim_{n \rightarrow \infty} n^{-2} \mathbb{V}\text{ar } A_n = v_0 \det \Sigma.$$

**Proof.** From (13) and Lemma A.1(ii), for  $p > 2$  we have  $\sup_n \mathbb{E}[(n^{-1}L_n^2)^{p/2}] < \infty$ . Hence  $n^{-1}L_n^2$  is uniformly integrable, and we deduce convergence of  $n^{-1}\mathbb{V}\text{ar } L_n$  in Corollary 2.6. Similarly, given  $\mathbb{E}[\|Z_1\|^p] < \infty$  for  $p > 4$ , Lemma 3.2(ii) shows that  $\mathbb{E}[A_n^{2(p/4)}] = O(n^{p/2})$ , so that  $\mathbb{E}[(n^{-2}A_n^2)^{p/4}]$  is uniformly bounded. Hence  $n^{-2}A_n^2$  is uniformly integrable, and we deduce convergence of  $n^{-2}\mathbb{V}\text{ar } A_n$  in Corollary 2.6.  $\square$

For the case with drift, we have the following variance result.

**Proposition 3.6.** Suppose that  $(M_p)$  holds for some  $p > 4$  and  $\mu \neq 0$ . Then

$$\lim_{n \rightarrow \infty} n^{-3} \mathbb{V}ar A_n = v_+ \|\mu\|_{\mu_\perp}^2.$$

**Proof.** Given  $\mathbb{E}[\|Z_1\|^p] < \infty$  for some  $p > 4$ , Lemma 3.2(i) shows that  $\mathbb{E}[A_n^{2(p/4)}] = O(n^{3p/4})$ , so that  $\mathbb{E}[(n^{-3}A_n^2)^{p/4}]$  is uniformly bounded. Hence  $n^{-3}A_n^2$  is uniformly integrable, so Corollary 2.8 yields the result.  $\square$

### 3.3. Variance bounds

The next result gives bounds on the quantities defined in (19).

**Proposition 3.7.** We have  $u_0(\Sigma) = 0$  if and only if  $\text{tr } \Sigma = 0$ . The following inequalities for the quantities defined at (19) hold.

$$\frac{263}{1080} \pi^{-3/2} e^{-144/25} \text{tr } \Sigma \leq u_0(\Sigma) \leq \frac{\pi^2}{2} \text{tr } \Sigma; \quad (20)$$

$$0 < \frac{4}{49} \left( e^{-7\pi^2/12} - \frac{1}{3} e^{-21\pi^2/4} \right)^2 \leq v_0 \leq 16(\log 2)^2 - \frac{\pi^2}{4}; \quad (21)$$

$$0 < \frac{2}{225} \left( e^{-25\pi/9} - \frac{1}{3} e^{-25\pi} \right) \leq v_+ \leq 4 \log 2 - \frac{2\pi}{9}. \quad (22)$$

Finally, if  $\Sigma = I$  we have the following sharper form of the lower bound in (20):

$$\mathbb{V}ar \ell_1 = u_0(I) \geq \frac{2}{5} \left( 1 - \frac{8}{25\pi} \right) e^{-25\pi/16} > 0.$$

For the proof of this result, we rely on a few facts about one-dimensional Brownian motion, including the bound (see e.g. equation (2.1) of [11]), valid for all  $r > 0$ ,

$$\mathbb{P} \left[ \sup_{0 \leq s \leq 1} |w(s)| \leq r \right] \geq \frac{4}{\pi} \left( e^{-\pi^2/(8r^2)} - \frac{1}{3} e^{-9\pi^2/(8r^2)} \right). \quad (23)$$

We let  $\Phi$  denote the distribution function of a standard normal random variable; we will also need the standard Gaussian tail bound (see e.g. [3, p. 12])

$$1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2} dy \geq \frac{1}{x\sqrt{2\pi}} \left( 1 - \frac{1}{x^2} \right) e^{-x^2/2}, \quad \text{for } x > 0. \quad (24)$$

We also note that for  $e \in \mathbb{S}_1$  the diffusion  $e \cdot (\Sigma^{1/2}b)$  is one-dimensional Brownian motion with variance parameter  $e^\top \Sigma e$ .

The idea behind the variance lower bounds is elementary. For a random variable  $X$  with mean  $\mathbb{E}X$ , we have, for any  $\theta \geq 0$ ,  $\mathbb{V}ar X = \mathbb{E}[(X - \mathbb{E}X)^2] \geq \theta^2 \mathbb{P}[|X - \mathbb{E}X| \geq \theta]$ . If  $\mathbb{E}X \geq 0$ , taking  $\theta = \alpha \mathbb{E}X$  for  $\alpha > 0$ , we obtain

$$\mathbb{V}ar X \geq \alpha^2 (\mathbb{E}X)^2 (\mathbb{P}[X \leq (1 - \alpha)\mathbb{E}X] + \mathbb{P}[X \geq (1 + \alpha)\mathbb{E}X]), \quad (25)$$

and our lower bounds use whichever of the latter two probabilities is most convenient.



**Proof of Proposition 3.7.** We start with the upper bounds. Snyder and Steele's bound (2) with the statement for  $\mathbb{V}\text{ar } L_n$  in Proposition 3.5 gives the upper bound in (20).

Bounding  $\tilde{a}_1$  by the area of a rectangle, we have

$$\tilde{a}_1 \leq r_1 \leq 2 \sup_{0 \leq s \leq 1} |w(s)|, \quad \text{a.s.}, \quad (26)$$

where  $r_1 := \sup_{0 \leq s \leq 1} w(s) - \inf_{0 \leq s \leq 1} w(s)$ . A result of Feller [7] states that  $\mathbb{E}[r_1^2] = 4 \log 2$ . So by the first inequality in (26), we have  $\mathbb{E}[\tilde{a}_1^2] \leq 4 \log 2$ , and by Proposition 3.4 we have  $\mathbb{E}\tilde{a}_1 = \frac{1}{3}\sqrt{2\pi}$ ; the upper bound in (22) follows.

Similarly, for any orthonormal basis  $\{e_1, e_2\}$  of  $\mathbb{R}^2$ , we bound  $a_1$  by a rectangle

$$a_1 \leq \left( \sup_{0 \leq s \leq 1} e_1 \cdot b(s) - \inf_{0 \leq s \leq 1} e_1 \cdot b(s) \right) \left( \sup_{0 \leq s \leq 1} e_2 \cdot b(s) - \inf_{0 \leq s \leq 1} e_2 \cdot b(s) \right),$$

and the two (orthogonal) components are independent, so  $\mathbb{E}[a_1^2] \leq (\mathbb{E}[r_1^2])^2 = 16(\log 2)^2$ , which with the fact that  $\mathbb{E}a_1 = \frac{\pi}{2}$  [4] gives the upper bound in (21).

We now move on to the lower bounds. Let  $e_\Sigma \in \mathbb{S}_1$  denote an eigenvector of  $\Sigma$  corresponding to the principal eigenvalue  $\lambda_\Sigma$ . Then since  $\Sigma^{1/2}h_1$  contains the line segment from 0 to any (other) point in  $\Sigma^{1/2}h_1$ , we have from monotonicity of  $\mathcal{L}$  that

$$\mathcal{L}(\Sigma^{1/2}h_1) \geq 2 \sup_{0 \leq s \leq 1} \|\Sigma^{1/2}b(s)\| \geq 2 \sup_{0 \leq s \leq 1} \left( e_\Sigma \cdot (\Sigma^{1/2}b(s)) \right).$$

Here  $e_\Sigma \cdot (\Sigma^{1/2}b)$  has the same distribution as  $\lambda_\Sigma^{1/2}w$ . Hence, for  $\alpha > 0$ ,

$$\begin{aligned} \mathbb{P} \left[ \mathcal{L}(\Sigma^{1/2}h_1) \geq (1 + \alpha) \mathbb{E} \mathcal{L}(\Sigma^{1/2}h_1) \right] &\geq \mathbb{P} \left[ \sup_{0 \leq s \leq 1} w(s) \geq \frac{1 + \alpha}{2} \lambda_\Sigma^{-1/2} \mathbb{E} \mathcal{L}(\Sigma^{1/2}h_1) \right] \\ &\geq \mathbb{P} \left[ \sup_{0 \leq s \leq 1} w(s) \geq 2(1 + \alpha)\sqrt{2} \right], \end{aligned}$$

using the fact that  $\lambda_\Sigma \geq \frac{1}{2} \text{tr } \Sigma$  and the upper bound in (12). Applying (25) to  $X = \mathcal{L}(\Sigma^{1/2}h_1) \geq 0$  gives, for  $\alpha > 0$ ,

$$\begin{aligned} \mathbb{V}\text{ar } \mathcal{L}(\Sigma^{1/2}h_1) &\geq \alpha^2 (\mathbb{E} \mathcal{L}(\Sigma^{1/2}h_1))^2 \mathbb{P} \left[ \sup_{0 \leq s \leq 1} w(s) \geq 2(1 + \alpha)\sqrt{2} \right] \\ &\geq \frac{32}{\pi} \alpha^2 (\text{tr } \Sigma) \left( 1 - \Phi(2(1 + \alpha)\sqrt{2}) \right), \end{aligned}$$

using the lower bound in (12) and the fact that  $\mathbb{P}[\sup_{0 \leq s \leq 1} w(s) \geq r] = 2\mathbb{P}[w(1) \geq r] = 2(1 - \Phi(r))$  for  $r > 0$ , which is a consequence of the reflection principle. Numerical curve sketching suggests that  $\alpha = 1/5$  is close to optimal; this choice of  $\alpha$  gives, using (24),

$$\mathbb{V}\text{ar } \mathcal{L}(\Sigma^{1/2}h_1) \geq \frac{32}{25\pi} (\text{tr } \Sigma) \left( 1 - \Phi(12\sqrt{2}/5) \right) \geq \frac{263}{1080} \pi^{-3/2} (\text{tr } \Sigma) \exp \left\{ -\frac{144}{25} \right\},$$

which is the lower bound in (20). We get a sharper result when  $\Sigma = I$  and  $\mathcal{L}(h_1) = \ell_1$ , since we know  $\mathbb{E}\ell_1 = \sqrt{8\pi}$  explicitly. Then, similarly to above, we get

$$\mathbb{V}\text{ar } \ell_1 \geq 8\pi\alpha^2 \mathbb{P} \left[ \sup_{0 \leq s \leq 1} w(s) \geq (1 + \alpha)\sqrt{2\pi} \right], \quad \text{for } \alpha > 0,$$

which at  $\alpha = 1/4$  yields the stated lower bound.

For areas, tractable upper bounds for  $a_1$  and  $\tilde{a}_1$  are easier to come by than lower bounds, and thus we obtain a lower bound on the variance by showing the appropriate area has positive probability of being smaller than the corresponding mean.

Consider  $a_1$ ; recall  $\mathbb{E}a_1 = \frac{\pi}{2}$  [4]. Since, for any orthonormal basis  $\{e_1, e_2\}$  of  $\mathbb{R}^2$ ,

$$a_1 \leq \pi \sup_{0 \leq s \leq 1} \|b(s)\|^2 \leq \pi \sup_{0 \leq s \leq 1} |e_1 \cdot b(s)|^2 + \pi \sup_{0 \leq s \leq 1} |e_2 \cdot b(s)|^2,$$

using the fact that  $e_1 \cdot b$  and  $e_2 \cdot b$  are independent one-dimensional Brownian motions,

$$\mathbb{P}[a_1 \leq r] \geq \mathbb{P} \left[ \sup_{0 \leq s \leq 1} |w(s)|^2 \leq \frac{r}{2\pi} \right]^2, \quad \text{for } r > 0.$$

We apply (25) with  $X = a_1$  and  $\alpha \in (0, 1)$ , and set  $r = (1 - \alpha)\frac{\pi}{2}$  to obtain

$$\begin{aligned} \mathbb{V}\text{ar } a_1 &\geq \alpha^2 \frac{\pi^2}{4} \mathbb{P} \left[ \sup_{0 \leq s \leq 1} |w(s)| \leq \frac{\sqrt{1 - \alpha}}{2} \right]^2 \\ &\geq 4\alpha^2 \left( \exp \left\{ -\frac{\pi^2}{2(1 - \alpha)} \right\} - \frac{1}{3} \exp \left\{ -\frac{9\pi^2}{2(1 - \alpha)} \right\} \right)^2, \end{aligned}$$

by (23). Taking  $\alpha = 1/7$  is close to optimal, and gives the lower bound in (21).

For  $\tilde{a}_1$ , we apply (25) with  $X = \tilde{a}_1$  and  $\alpha \in (0, 1)$ . Using the fact that  $\mathbb{E}\tilde{a}_1 = \frac{1}{3}\sqrt{2\pi}$  (from Proposition 3.4) and the weaker of the two bounds in (26), we obtain

$$\begin{aligned} \mathbb{V}\text{ar } \tilde{a}_1 &\geq \alpha^2 \frac{2\pi}{9} \mathbb{P} \left[ \sup_{0 \leq s \leq 1} |w(s)| \leq \frac{(1 - \alpha)\sqrt{2\pi}}{6} \right] \\ &\geq \frac{8}{9}\alpha^2 \left( \exp \left\{ -\frac{9\pi}{4(1 - \alpha)^2} \right\} - \frac{1}{3} \exp \left\{ -\frac{81\pi}{4(1 - \alpha)^2} \right\} \right), \end{aligned}$$

by (23). Taking  $\alpha = 1/10$  is close to optimal, and gives the lower bound in (22).  $\square$

**Remark.** The main interest of the lower bounds in Proposition 3.7 is that they are *positive*; they are certainly not sharp. The bounds can surely be improved, although the authors have been unable to improve any of them sufficiently to warrant reporting the details here. We note just the following idea. A lower bound for  $\tilde{a}_1$  can be obtained by conditioning on  $\theta := \sup\{s \in [0, 1] : w(s) = 0\}$  and using the fact that the maximum of  $w$  up to time  $\theta$  is distributed as the maximum of a scaled Brownian bridge; combining this with the previous argument improves the lower bound on  $v_+$  to  $2.09 \times 10^{-6}$ .

## 4. Concluding remarks

### 4.1. Exact evaluation of limiting variances

It would, of course, be of interest to evaluate any of  $u_0$ ,  $v_0$ , or  $v_+$  exactly. In general this looks hard. Since the first version of this paper was written, the paper [18] came to the authors' attention; this provides a key component to a possible approach to evaluating  $u_0$ . By Cauchy's formula and Fubini's theorem,

$$\mathbb{E}[\ell_1^2] = \int_{\mathbb{S}_1} \int_{\mathbb{S}_1} \mathbb{E} \left[ \left( \sup_{0 \leq s \leq 1} (e_1 \cdot b(s)) \right) \left( \sup_{0 \leq t \leq 1} (e_2 \cdot b(t)) \right) \right] de_1 de_2.$$

Here, the two standard one-dimensional Brownian motions  $e_1 \cdot b$  and  $e_2 \cdot b$  have correlation determined by the cosine of the angle  $\phi$  between  $e_1$  and  $e_2$ , i.e.,

$$\mathbb{E}[(e_1 \cdot b(s))(e_2 \cdot b(t))] = (s \wedge t) e_1 \cdot e_2 = (s \wedge t) \cos \phi.$$

The result of Rogers and Shepp [18] then shows that

$$\mathbb{E} \left[ \left( \sup_{0 \leq s \leq 1} (e_1 \cdot b(s)) \right) \left( \sup_{0 \leq t \leq 1} (e_2 \cdot b(t)) \right) \right] = c(\cos \phi),$$

where the function  $c$  is given explicitly in [18]. Using this result, we obtain

$$\mathbb{E}[\ell_1^2] = 4\pi \int_{-\pi/2}^{\pi/2} c(\sin \theta) d\theta = 4\pi \int_{-\pi/2}^{\pi/2} d\theta \int_0^\infty du \cos \theta \frac{\cosh(u\theta)}{\sinh(u\pi/2)} \tanh \left( \frac{(2\theta + \pi)u}{4} \right).$$

We have not been able to deal with this integral analytically, but numerical integration gives  $\mathbb{E}[\ell_1^2] \approx 26.1677$ , which with the fact that  $\mathbb{E}\ell_1 = \sqrt{8\pi}$  gives  $u_0(I) = \mathbb{V}\text{ar } \ell_1 \approx 1.0350$ , in reasonable agreement with the simulation estimate in Table 1.

Another possible approach to evaluating  $u_0$  is suggested by a remarkable computation of Goldman [8] for the analogue of  $u_0(I) = \mathbb{V}\text{ar } \ell_1$  for the planar *Brownian bridge*. Specifically, if  $b'_t$  is the standard Brownian bridge in  $\mathbb{R}^2$  with  $b'_0 = b'_1 = 0$ , and  $\ell'_1 = \mathcal{L}(\text{hull } b'[0, 1])$  the perimeter length of its convex hull, [8, Théorème 7] states that

$$\mathbb{V}\text{ar } \ell'_1 = \frac{\pi^2}{6} \left( 2\pi \int_0^\pi \frac{\sin \theta}{\theta} d\theta - 2 - 3\pi \right) \approx 0.34755.$$

### 4.2. Final comments

The framework of Section 2 shows that whenever a discrete-time process in  $\mathbb{R}^d$  converges weakly to a limit on the space of continuous paths, the corresponding convex hulls converge. It would be of interest to extend the framework to admit discontinuous limit processes, such as Lévy processes with jumps [13] that arise as scaling limits of random walks whose increments have infinite variance.

The idea used in the proof of Proposition 3.4, first establishing the existence of a limit for a class of models and then choosing a particular model for which the limit can be conveniently evaluated, goes back at least to Kac; see [5, p. 293].

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## Appendix. Random walk norms

**Lemma A.1.** *Let  $p > 1$ . Suppose that  $\mathbb{E}[\|Z_1\|^p] < \infty$ .*

- (i) *For any  $e \in \mathbb{S}_1$  such that  $e \cdot \mu = 0$ ,  $\mathbb{E}[\max_{0 \leq m \leq n} |S_m \cdot e|^p] = O(n^{1 \vee (p/2)})$ .*
- (ii) *Moreover, if  $\mu = 0$ , then  $\mathbb{E}[\max_{0 \leq m \leq n} \|S_m\|^p] = O(n^{1 \vee (p/2)})$ .*
- (iii) *On the other hand, if  $\mu \neq 0$ , then  $\mathbb{E}[\max_{0 \leq m \leq n} |S_m \cdot \hat{\mu}|^p] = O(n^p)$ .*

**Proof.** Given that  $\mu \cdot e = 0$ ,  $S_n \cdot e$  is a martingale, and hence, by convexity,  $|S_n \cdot e|$  is a non-negative submartingale. Then, for  $p > 1$ ,

$$\mathbb{E} \left[ \max_{0 \leq m \leq n} |S_m \cdot e|^p \right] \leq \left( \frac{p}{p-1} \right)^p \mathbb{E} [|S_n \cdot e|^p] = O(n^{1 \vee (p/2)}),$$

where the first inequality is Doob's  $L^p$  inequality [10, p. 505] and the second is the Marcinkiewicz–Zygmund inequality [10, p. 151]. This gives part (i).

Part (ii) follows from part (i): take  $\{e_1, e_2\}$  an orthonormal basis of  $\mathbb{R}^2$  and apply (i) with each basis vector; (ii) then follows from the triangle inequality  $\max_{0 \leq m \leq n} \|S_m\| \leq \max_{0 \leq m \leq n} |S_m \cdot e_1| + \max_{0 \leq m \leq n} |S_m \cdot e_2|$  together with Minkowski's inequality.

Part (iii) follows from the fact that  $\max_{0 \leq m \leq n} |S_m \cdot \hat{\mu}| \leq \sum_{k=1}^n |Z_k \cdot \hat{\mu}| \leq \sum_{k=1}^n \|Z_k\|$  and an application of Rosenthal's inequality [10, p. 151] to the latter sum.  $\square$

## References

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